TOPOLOGY III - BACKPAPER EXAM.

Time: 3 hours Max. marks: 100

Answer all questions. You may use results proved in class after correctly quoting them. Any other claim must be accompanied by a proof.

(1) Let X be a space and let $S_*(X)$ denote its singular chain complex. For an abelian group G, define
$$h_n(X;G) = H_n(\mathrm{Hom}(G, S_*(X))$$
where $\mathrm{Hom}(G, S_*(X))$ denotes the chain complex whose n-th chain group is $\mathrm{Hom}(G, S_n(X))$ with he obvious boundary map. Compute the groups $h_n(X;G)$ when $G = \mathbb{Z}, \mathbb{Z}_2, \mathbb{Q}$. [6]

(2) State the Kunneth theorem for singular homology.
  (a) Compute $\mathrm{Tor}(\mathbb{Z}_2, \mathbb{Z})$, $\mathrm{Tor}(\mathbb{Z}, \mathbb{Z}_2)$, $\mathrm{Tor}(\mathbb{Z}_2, \mathbb{Z}_3)$.
  (b) Compute $H_i(\mathbb{R}P^2 \times \mathbb{R}P^3; G)$ for $G = \mathbb{Z}_2, \mathbb{Z}, \mathbb{Q}$. [4+8 +10]

(3) Define the notion of the degree of a map between two connected closed oriented manifolds.
  (a) Show that if $f, g : X \longrightarrow Y$ are homotopic maps between two closed oriented manifolds, then they have the same degree.
  (b) Show that if X is a connected closed orientable n-manifold, then there exists a map $f : X \longrightarrow S^n$ of degree 1.
  (c) Construct a map $g : S^2 \longrightarrow S^2$ of degree 2. [2+4+10+ 6=22]

(4) Let X be a connected closed orientable n-manifold. Assume that there exists a map $f : S^n \longrightarrow X$ of degree $k \neq 0$. Prove that $H_i(X;\mathbb{Q}) \cong H_i(S^n;\mathbb{Q})$. [12]

(5) Let $(X, x_0), (Y, y_0)$ be based spaces with X, Y locally compact Hausdorff. Prove that there is bijection
$$[\Sigma(X, x_0), (Y, y_0)] \longrightarrow [(X, x_0), \Omega(Y, y_0)]$$
between the homotopy sets. [12]

(6) Define the term : fibration. Show that the projection
$$\{(x, y) \in \mathbb{R}^2 \, : \, x \in [0, 1], y \leq 1 - x\} \longrightarrow [0, 1]$$
to the first factor is a fibration. [2+6]

(7) Define the term: fiber bundle. Show that there are fiber bundles
$$S^1 \hookrightarrow S^{2n+1} \longrightarrow \mathbb{C}P^n, \quad SO(n-1) \hookrightarrow SO(n) \longrightarrow S^{n-1}$$
Use the above to compute
  (a) $\pi_i(\mathbb{C}P^n)$ (in terms of those of the homotopy groups of the spheres) and
  (b) $\pi_i(SO(n))$ for $i = 1, 2$ ($n \geq 2$). [2 + 8+8]